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QUALITY FACTOR FOR ANTENNAS (A TUTORIAL)

Arthur D. Yaghjian (EM Consultant, USA) a.yaghjian@comcast.net
Mats Gustafsson (Lund U., Sweden)
B. Lars G. Jonsson (KTH, Sweden)

Reference
"Minimum Q for Lossy and Lossless Electrically Small Dipole Antennas," *PIER*, 143, 2013

Acknowledgements
Steven Best, Howard Stuart, Olav Breinbjerg, Alex Kim

Abstract

Beginning with an introduction to bandwidth and Q of electrically small antennas (ESA's), general formulas are given for energy density and Q in highly dispersive lossy (HDL) material and low-loss or nondispersive conductive (nc) material. Lower-bound formulas for the Q of electrically small dipole antennas are found in terms of their volume polarizabilities, radiated powers, and radiation efficiencies. It is shown that highly dispersive materials enable single dipole antennas to overcome the Chu lower bound by sacrificing efficiency. The talk ends with a hypothetical ESA that has a Q equal to half the "Chu lower bound".

WHY TALK ABOUT *Q* IF IT'S THE ANTENNA BANDWIDTH THAT MATTERS?

- **2/*Q* equals the half power fractional bandwidth.**
- ***Q* is computed at a single frequency.**
- **Lower bounds on *Q* exist for electrically small antennas with a given shape and volume *V*.**
- ***Q* is defined in terms of the antenna's *E* & *H* fields and the material constitutive parameters, and thus can give insight into how to lower the *Q*.**
- ***Q* leads to general theorems on the fundamental limitations of antennas.**

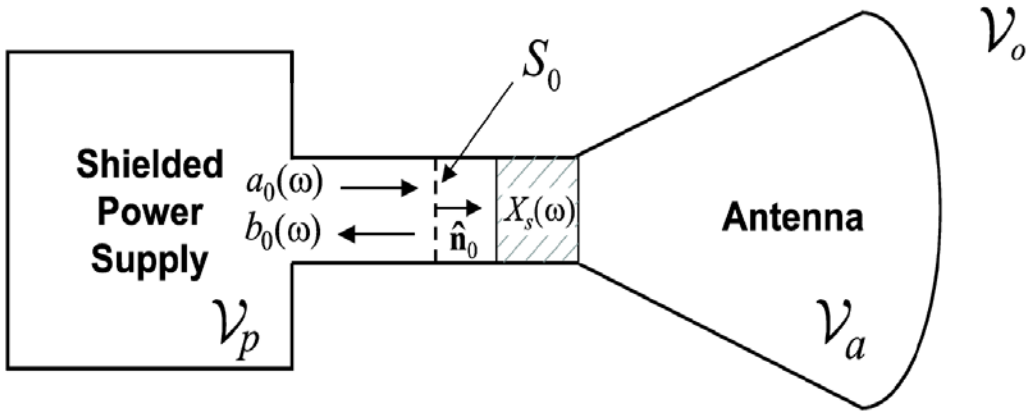
OUTLINE OF TALK

- Begin with an introduction to bandwidth and Q of electrically small antennas (ESA's).
- Show general formulas for energy density and Q in highly dispersive lossy (HDL) material (and low-loss or nondispersive conductive (nc) material).
- Find simple formulas for Q lower bounds of electrically small dipole antennas in terms of their volume polarizabilities, radiated powers, and radiation efficiencies.
- Give a hypothetical ESA that has a Q equal to half the "Chu lower bound".

PRELIMINARY REMARKS

- Bandwidth is defined relative to a change in the input reflection coefficient of the mode feeding the antenna (not, for example, relative to far-field degradation).
- We ignore any extra internal tuning that may be required to maintain a desired phase difference between the electric and magnetic dipole moments of the antenna.
- We consider transmitting antennas only. (The bandwidth of receiving antennas is determined mainly by the signal to noise ratio of the amplifier output.)
- Concentrate on Q lower bounds for ES dipole antennas and not on optimization of Q for larger antennas using current integration (Gustafsson, Vandenbosch).

MATCHED VSWR BW FOR TUNED ANTENNA



One-port, linear, passive, antennas tuned to an isolated resonance (or antiresonance) with a series inductor or capacitor, considered part of the antenna proper.

The bandwidth of an antenna can be increased without lowering the Q by using:

- **Multiple resonances (Bode-Fano impedance matching)**
- **Potential bandwidth (unmatched at center frequency)**
- **Non-Foster matching networks (active, nonlinear)**

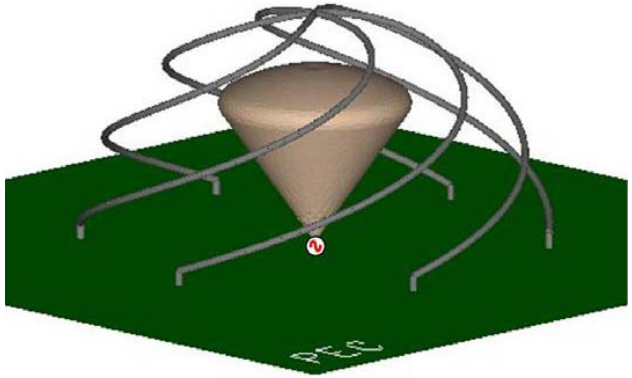
USEFUL TO KNOW THE Q LOWER BOUND

- $FBW_{HP} = 2 / Q$ (single isolated resonance)
- $FBW \approx -\pi / [\ln(\Gamma)\, Q]$ ($\approx 9 / Q$ for HPBW)
  (multiple resonances on antennas with $ka < .3$)
  This upper-bound on FBW is estimated using infinite-stage Bode-Fano matching. Typically half this bandwidth is obtained in practice.
- FBW can sometimes be increased by not perfectly matching to a center frequency ("potential FBW" increase as much as 50%).

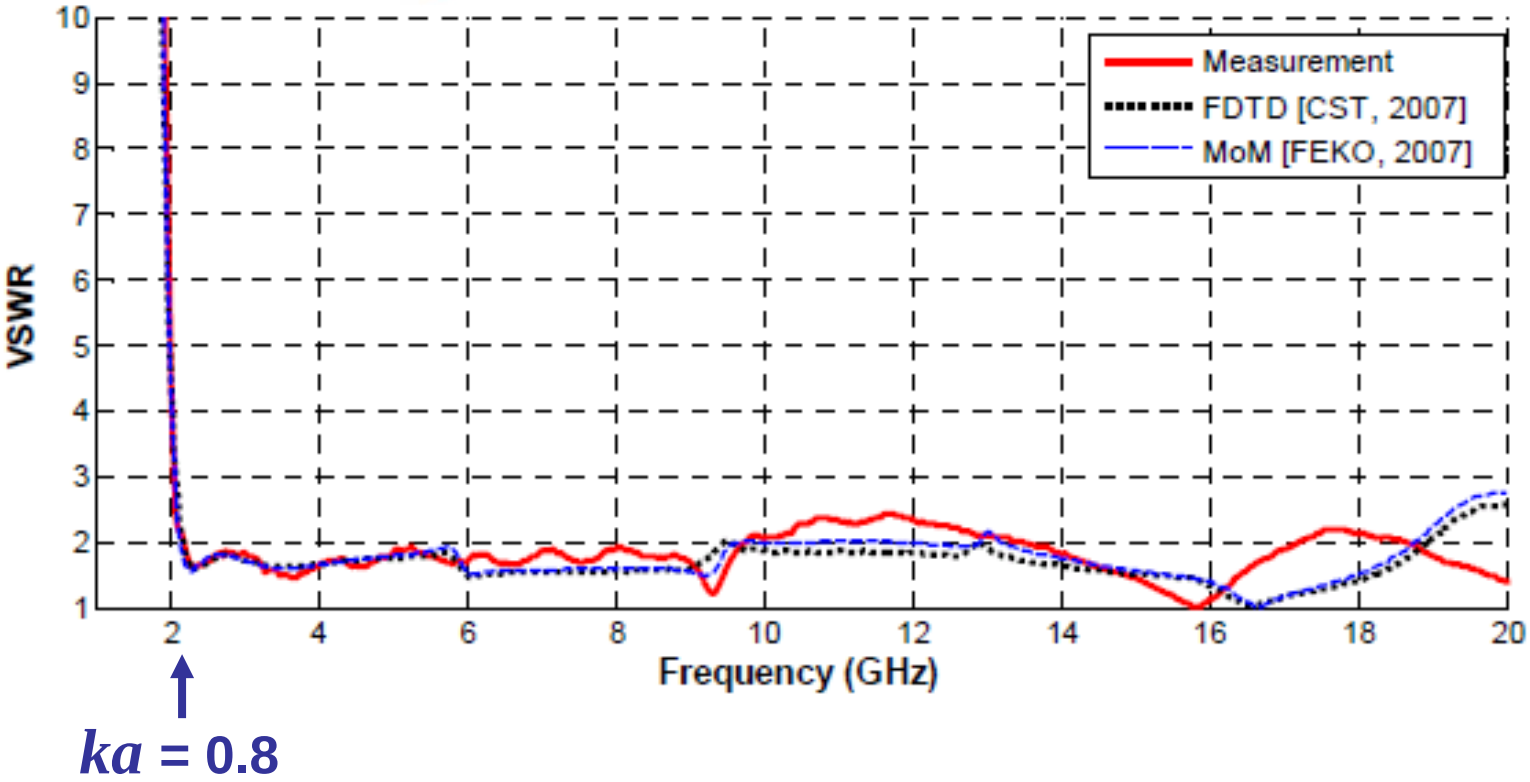
Non-Foster matching to increase BW is not part of this talk.

ULTRA-WIDEBAND ANTENNA FOR $ka > 0.8$

(Of course, the pattern changes over the BW)

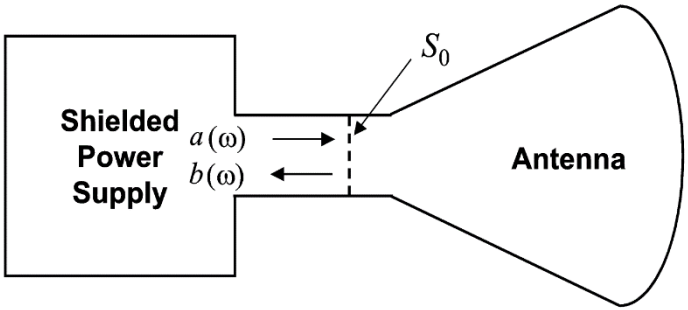


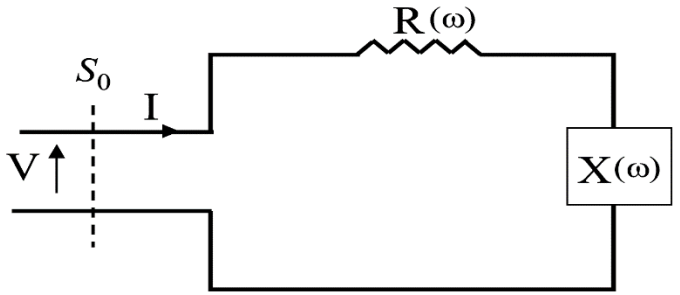
Yang, Davis & Stutzman
URSI GA, 2008

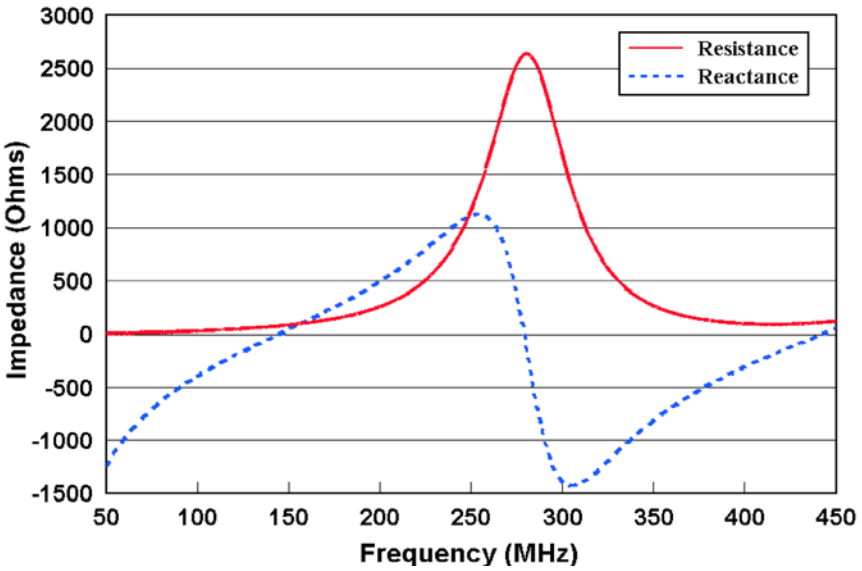


FOSTER REACTANCE THEOREM DOES NOT APPLY TO ANTENNA IMPEDANCE

$$(e^{j\omega t},\ \omega = 2\pi f > 0)$$



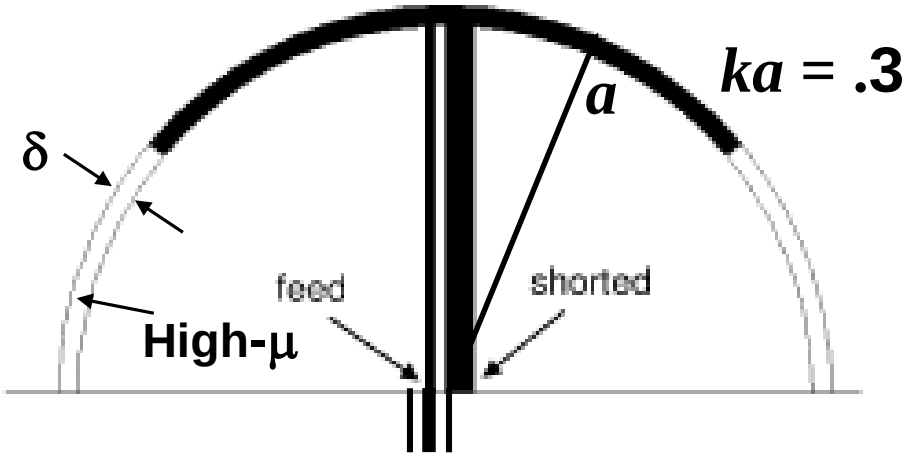




Foster Reactance Theorem does *not* apply to "lossless" antennas because of the radiation resistance.

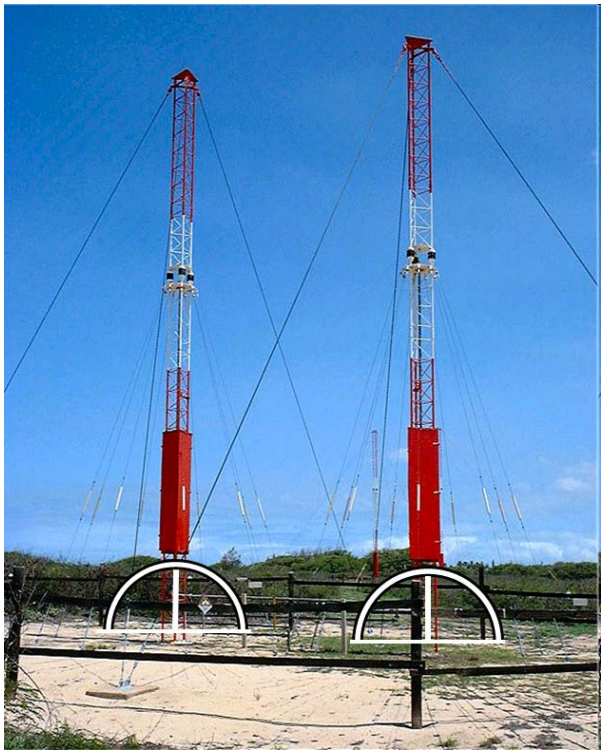
CHU LB CAN BE APPROACHED FOR ELECTRIC DIPOLE ANTENNAS

Previous antenna designs and Thal's 2006 paper (*IEEE APS Trans.*) led to the consensus (although Thal did not say this) that it is impossible to design an electrically small electric dipole antenna with a Q less than 1.5 times the Chu lower bound $(ka)^{-3}$ --- because magnetic charge/current does not exist.

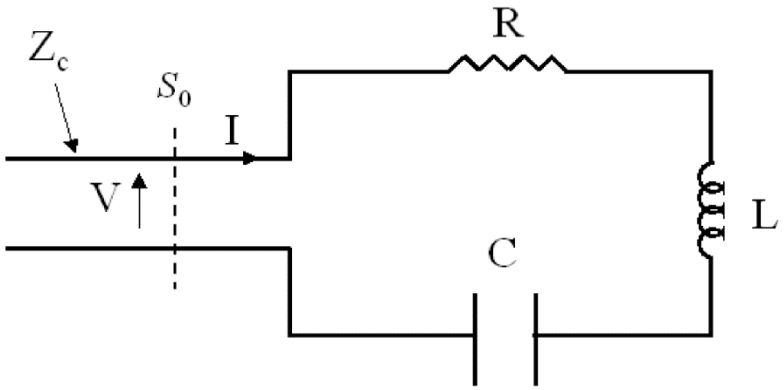


Electric dipole antenna ($Q=1.1Q_{Chu}$)
Stuart & Yaghjian, 2010

(High-μ coated PEC shell, O. Kim, 2011)



MATCHED VSWR BANDWIDTH FOR RESONANT SERIES RLC CIRCUIT



$$\Gamma = \frac{Z - Z_c}{Z + Z_c}, \quad Z = R + j\left(\omega L - \frac{1}{\omega C}\right),$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

With $Z_c = R$, find $\omega_\pm$ such that

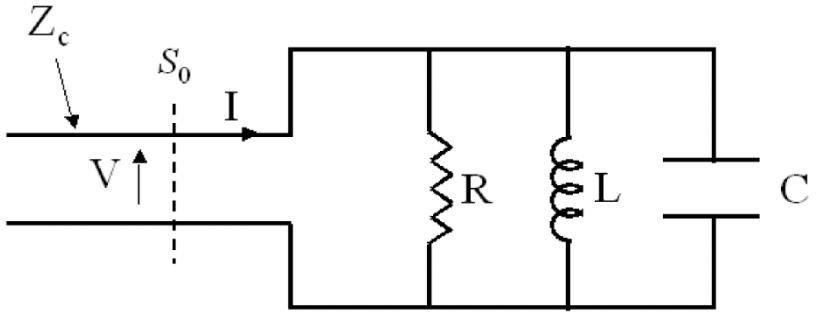
$$|\Gamma(\omega)|^2 = \frac{[\omega L - 1/(\omega C)]^2}{4R^2 + [\omega L - 1/(\omega C)]^2} = \frac{1}{2}$$

$$FBW_{HP} = \frac{\omega_+ - \omega_-}{\omega_0} = 2R\sqrt{\frac{C}{L}}$$

$$Q = \frac{\omega_0 W(\omega_0)}{P_{\mathcal{L}}} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{2}{FBW_{HP}} = \frac{\omega_0 X'(\omega_0)}{2R}$$

J. Chu (1948), R.L. Fante (1969), *Et al.*

MATCHED VSWR BANDWIDTH FOR RESONANT PARALLEL RLC CIRCUIT



$$\Gamma = \frac{1 - YZ_c}{1 + YZ_c}, \quad Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right),$$

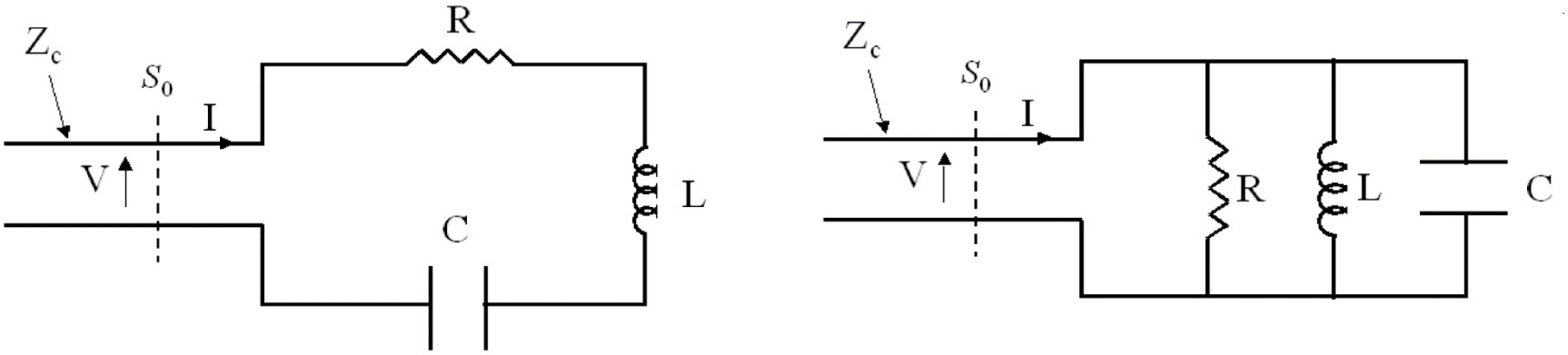
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

With $Z_c = R$, find $\omega_\pm$ such that

$$|\Gamma(\omega)|^2 = \frac{R^2[\omega L - 1/(\omega C)]^2}{4 + R^2[\omega L - 1/(\omega C)]^2} = \frac{1}{2} \qquad FBW_{HP} = \frac{\omega_+ - \omega_-}{\omega_0} = \frac{2}{R}\sqrt{\frac{L}{C}}$$

$$Q = \frac{\omega_0 W(\omega_0)}{P_{\mathcal{L}}} = R\sqrt{\frac{C}{L}} = \frac{2}{FBW_{HP}} = \frac{-\omega_0 X'(\omega_0)}{2R}$$

FOR RESONANT SERIES & PARALLEL RLC CIRCUITS



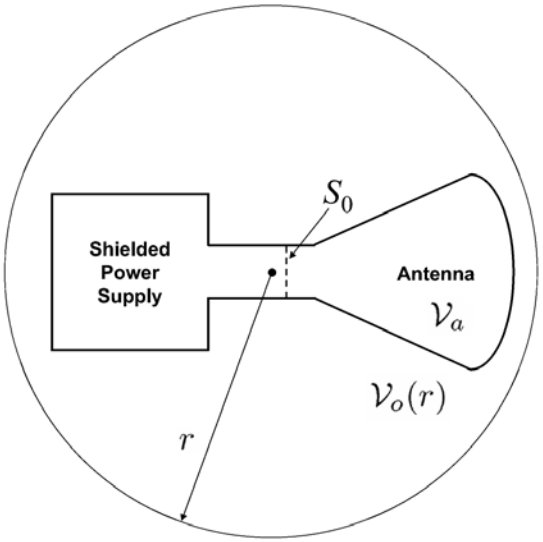
$$W = \frac{1}{4}CV_C^2 + \frac{1}{4}LI_L^2 = \frac{1}{4}\int_C \epsilon_0|E|^2 dV + \frac{1}{4}\int_L \mu_0|H|^2 dV$$

$$Q = \frac{\omega_0 W(\omega_0)}{P_{\mathcal{L}}} = \frac{2}{FBW_{HP}} = \frac{\omega_0|X'(\omega_0)|}{2R}$$

D.R. Rhodes (1976)

For antennas R, L, and C are functions of frequency and, in general, these formulas for Q and bandwidth are no longer accurate!

GENERAL FORMULA FOR IMPEDANCE BANDWIDTH OF ANTENNAS



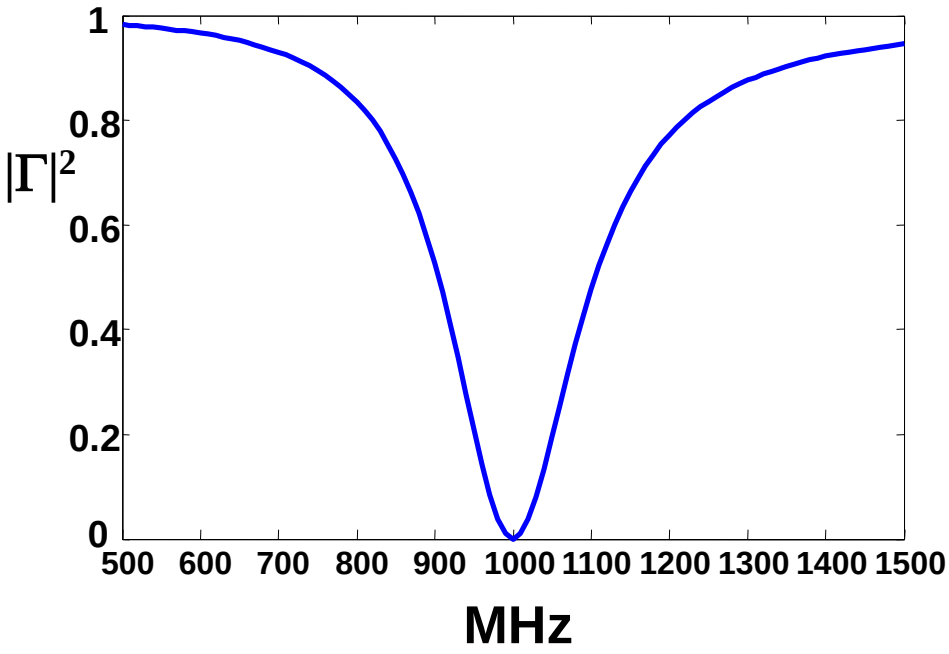
One-port, linear, passive, antenna tuned to resonance or antiresonance

$$FBW_V(\omega) \approx \frac{4\sqrt{\beta}R(\omega)}{\omega|Z'(\omega)|} \qquad \sqrt{\beta} = \frac{s-1}{2\sqrt{s}}$$

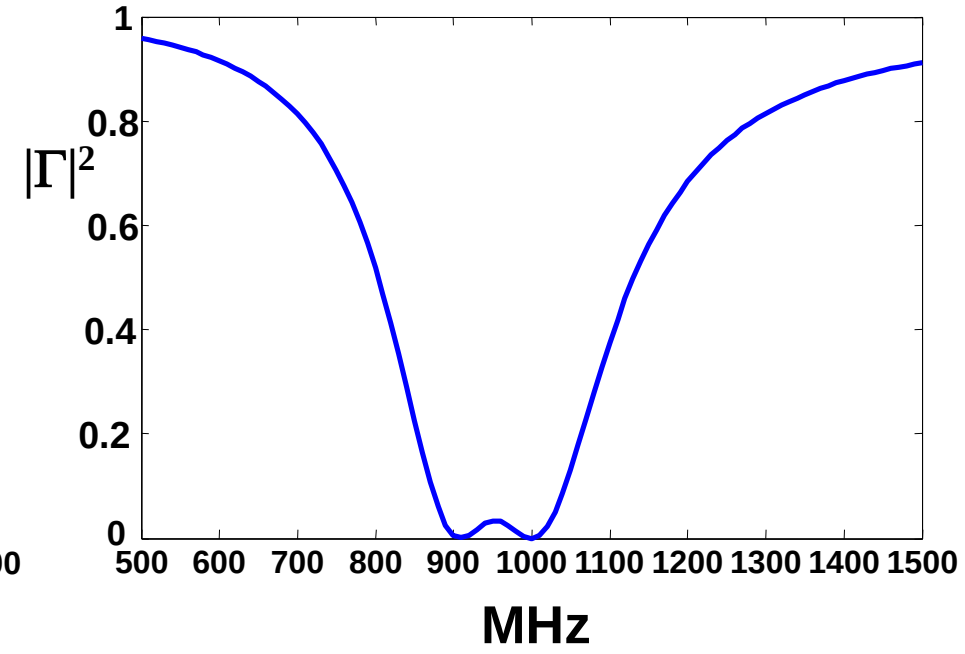
This formula is always an accurate approximation for the fractional VSWR bandwidth at any tuned frequency (ω) if β is chosen small enough that $Z'(\omega)$ does not change greatly over the bandwidth [Yaghjian & Best 2005].

IMPEDANCE BANDWIDTH FORMULA

$$FBW_V(\omega) \approx \frac{4\sqrt{\beta}R(\omega)}{\omega|Z'(\omega)|} \qquad \sqrt{\beta} = \frac{s-1}{2\sqrt{s}}$$



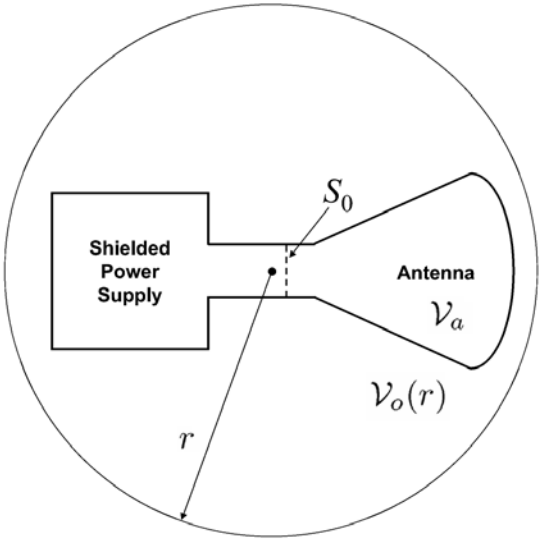
Single Resonance



Double Resonance

(|Z'| varies greatly over -3dB BW)

GENERAL FORMULA FOR IMPEDANCE BANDWIDTH OF ANTENNAS



One-port, linear, passive, antenna tuned to resonance or antiresonance

$$FBW_V(\omega) \approx \frac{4\sqrt{\beta}R(\omega)}{\omega|Z'(\omega)|} \qquad \sqrt{\beta} = \frac{s-1}{2\sqrt{s}}$$

Can we write *Z'* in terms of fields to get a universally valid formula for *Q*? Yes, but it requires several integrals of fields and frequency derivatives of fields that give little physical insight useful for antenna design.

ACCURATE FORMULA FOR THE *Q* OF NONDISPERSIVE CONDUCTIVE (nc) ESA's

$$Q(\omega) = \eta \frac{\omega W(\omega)}{P_R(\omega)} \approx \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$

(Quasi-static fields of ESA's)

$$W(\omega) \approx \frac{1}{4}\mathrm{Re}\int_{\mathcal{V}_\infty} \Big\{ \underbrace{\mathbf{E}^* \cdot (\omega\overline{\boldsymbol{\epsilon}})' \cdot \mathbf{E}}_{\text{Brillouin (1921)}} + \mathbf{H}^* \cdot (\omega\overline{\boldsymbol{\mu}})' \cdot \mathbf{H} + \left[\mathbf{E}^* \cdot (\omega\overline{\boldsymbol{\tau}})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega\overline{\boldsymbol{\nu}})' \cdot \mathbf{E}\right] \Big\} d\mathcal{V}$$

***W* is the internal energy supply by the fields to the charge carriers in a lossless bianisotropic material.**

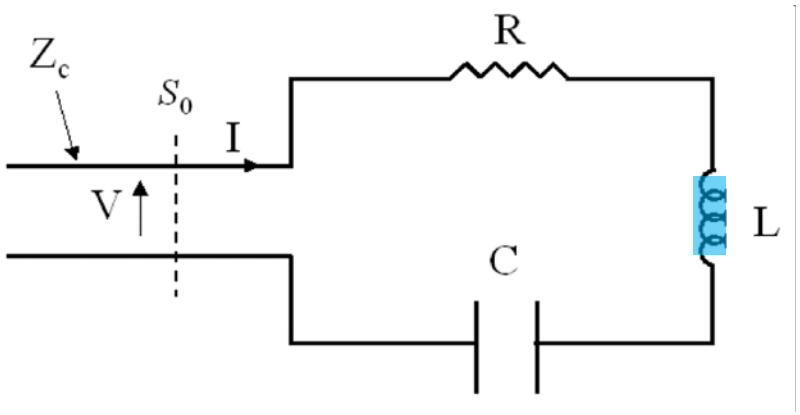
$$\mathbf{D}(\mathbf{r}) = \overline{\boldsymbol{\epsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \overline{\boldsymbol{\tau}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

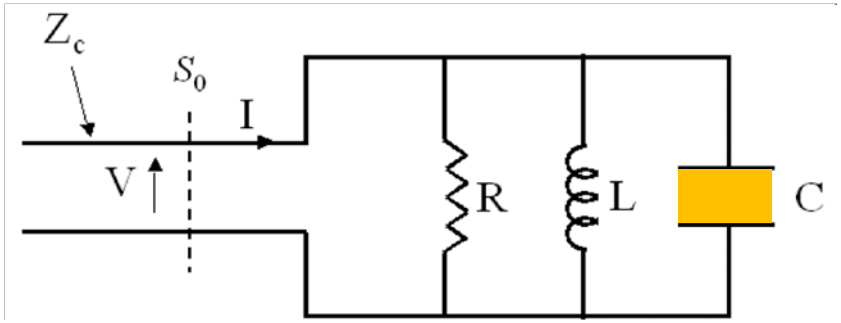
$$\mathbf{B}(\mathbf{r}) = \overline{\boldsymbol{\mu}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}) + \overline{\boldsymbol{\nu}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

ACCURATE FORMULA FOR THE Q OF NONDISPERSIVE CONDUCTIVE (nc) ESA's

$$Q(\omega) = \eta \frac{\omega W(\omega)}{P_R(\omega)} = \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$

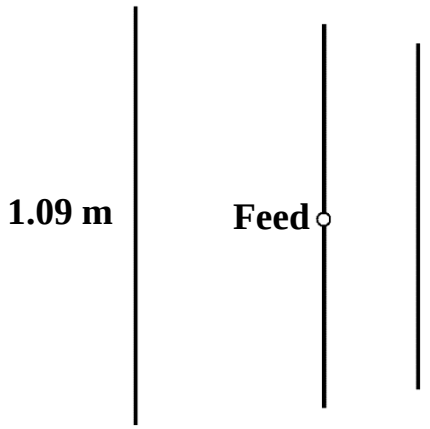
Holds exactly for RLC circuits with L and C filled with lossless (nc) scalar ε, μ, τ, ν media (bi-isotropic).



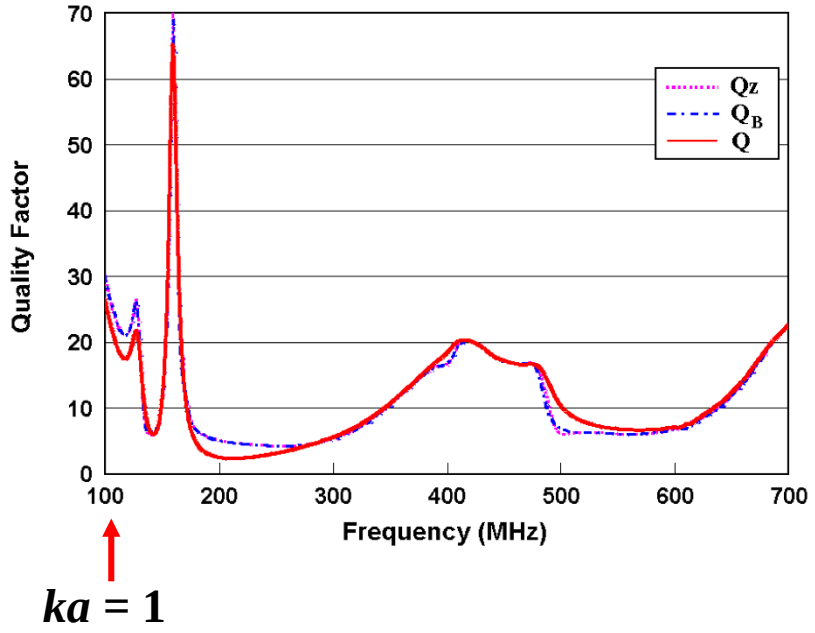


ACCURATE FORMULA FOR THE Q OF NONDISPERSIVE CONDUCTIVE (nc) ESA's

$$Q(\omega) = \eta \frac{\omega W(\omega)}{P_R(\omega)} \approx \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$



Three element Yagi



CAN WE FIND A GENERAL FORMULA FOR *Q* (*W*) IN HIGHLY DISPERSIVE LOSSY ANTENNAS?

General Consensus

"In the presence of dispersion and losses, the knowledge of the permittivity and permeability functions alone is insufficient to provide an expression for the **stored** electromagnetic energy density. This is because a very detailed model of the microstructured medium under investigation is needed. Unfortunately, this means that the problem of finding the energy density has to be solved separately for every material." *Boardman & Marinov, PRB, April 2006*

Therefore, we need to find an alternative electromagnetic energy density that generalizes the concept of "stored" energy in highly dispersive lossy antenna material.

CAN WE FIND A USEFUL GENERAL FORMULA FOR *Q* (*W*) IN HIGHLY DISPERSIVE LOSSY ANTENNAS?

$$Q(\omega) = \eta \frac{\omega W(\omega)}{P_R(\omega)} \approx \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$

(Quasi-static fields of ESA's)

$$W(\omega) \approx \frac{1}{4} \cancel{\mathrm{Re}} \int_{\mathcal{V}_\infty} \Big\{ \mathbf{E}^* \cdot (\omega \overline{\boldsymbol{\epsilon}})' \cdot \mathbf{E} + \mathbf{H}^* \cdot (\omega \overline{\boldsymbol{\mu}})' \cdot \mathbf{H} + \left[\mathbf{E}^* \cdot (\omega \overline{\boldsymbol{\tau}})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega \overline{\boldsymbol{\nu}})' \cdot \mathbf{E} \right] \Big\} d\mathcal{V}$$

$$\mathbf{D}(\mathbf{r}) = \overline{\boldsymbol{\epsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \overline{\boldsymbol{\tau}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

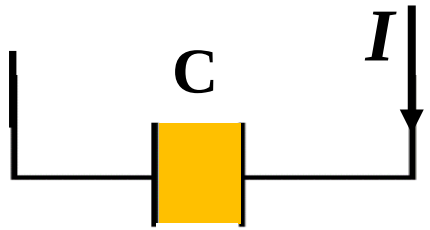
$$\mathbf{B}(\mathbf{r}) = \overline{\boldsymbol{\mu}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}) + \overline{\boldsymbol{\nu}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

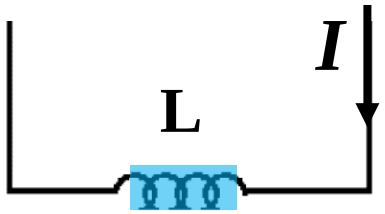
HDL *W* HOLDS EXACTLY FOR ε AND μ in C and L

$$|W(\omega)| = \frac{1}{4}|Z'(\omega)||I|^2$$

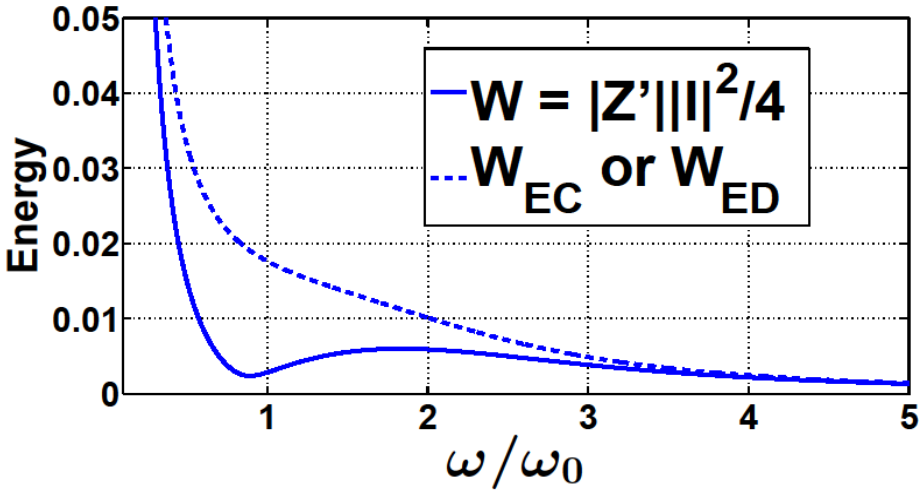
$$W(\omega) = \frac{1}{4}\int_C (\omega\epsilon)'|E|^2 dV$$

$$W(\omega) = \frac{1}{4}\int_L (\omega\mu)'|H|^2 dV$$

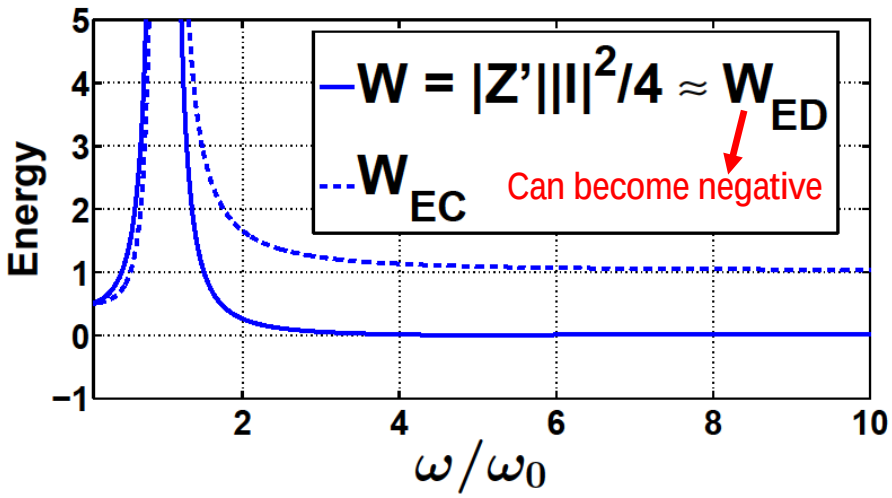




Lorentzian permittivity



Resonant permeability

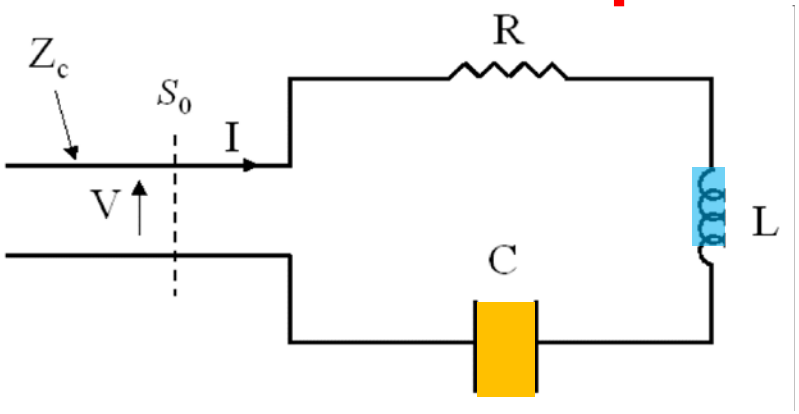


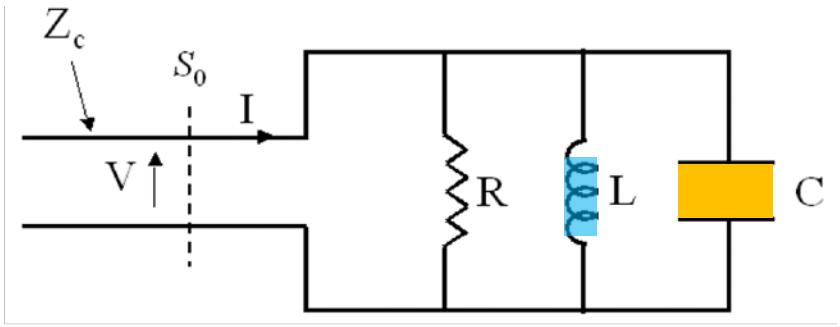
GENERAL FORMULA FOR HIGH-Q HIGHLY DISPERSIVE LOSSY ESA's

$$Q(\omega) = \eta \frac{\omega |W(\omega)|}{P_R(\omega)} \approx \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$

$$W(\omega) \approx \frac{1}{4} \int_{\mathcal{V}_\infty} \left\{ \mathbf{E}^* \cdot (\omega \bar{\epsilon})' \cdot \mathbf{E} + \mathbf{H}^* \cdot (\omega \bar{\mu})' \cdot \mathbf{H} + \left[\mathbf{E}^* \cdot (\omega \bar{\tau})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega \bar{\nu})' \cdot \mathbf{E} \right] \right\} d\mathcal{V}$$

Verified for RLC circuits with L and C filled with HDL scalar permittivity and permeability.





MINIMUM Q FOR ES DIPOLE ANTENNAS

$$Q(\omega) \approx \frac{\eta\,\omega}{4(P_e+P_m)}\left|\int_{\mathcal{V}_a}\Big\{\mathbf{E}^*\cdot(\omega\overline{\boldsymbol{\epsilon}})'\cdot\mathbf{E}+\mathbf{H}^*\cdot(\omega\overline{\boldsymbol{\mu}})'\cdot\mathbf{H} + [\mathbf{E}^*\cdot(\omega\overline{\boldsymbol{\tau}})'\cdot\mathbf{H}+\mathbf{H}^*\cdot(\omega\overline{\boldsymbol{\nu}})'\cdot\mathbf{E}]\Big\}\,d\mathcal{V} + \int_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2+\mu_0|\mathbf{H}_m|^2\right)d\mathcal{V}\right|$$

Antenna

$\mathcal{V}_a$

Equivalent surface currents

$$\mathbf{K}_e=\hat{\mathbf{n}}\times\mathbf{H}\,,\qquad \mathbf{K}_m=-\hat{\mathbf{n}}\times\mathbf{E}$$

$$\mathrm{Re}\int_{\mathcal{V}_a}[\mathbf{H}\cdot\overline{\boldsymbol{\mu}}^*\cdot\mathbf{H}^*-\mathbf{E}\cdot\overline{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}^*+\mathbf{H}\cdot(\overline{\boldsymbol{\nu}}^*-\overline{\boldsymbol{\tau}}_t)\cdot\mathbf{E}^*]\,d\mathcal{V} \approx \int_{\mathcal{V}_{out}}\left(\epsilon_0|\mathbf{E}_e|^2-\mu_0|\mathbf{H}_m|^2\right)d\mathcal{V}$$

MINIMUM Q FOR ES DIPOLE ANTENNAS

Antenna V_a

$$Q_{lb}^{hdl}(\omega) = \eta\,\omega\,\mathrm{Min}\left[\frac{\int_{\mathcal{V}_{out}} \left(\epsilon_0|\mathbf{E}_e|^2 + \mu_0|\mathbf{H}_m|^2\right) d\mathcal{V}}{4(P_e + P_m)}\right]$$

$$Q_{lb}^{hdl}(\omega) = \frac{3\pi\eta}{k^3}\left[\frac{1}{\alpha_e - \alpha_m}\right]$$

$$Q_{lb}^{hdl} = .5\eta/(ka)^3$$

spherical electric or magnetic dipole

Antenna V_a

$$Q_{lb}^{nc}(\omega) = 2\eta\omega\mathrm{Min}\left[\frac{\int_{\mathcal{V}_{out}} \left[\epsilon_0|\mathbf{E}_e|^2, \mu_0|\mathbf{H}_m|^2\right] d\mathcal{V}}{4(P_e + P_m)}\right]$$

$$Q_{lb}^{nc}(\omega) = \frac{6\pi\eta}{k^3(1+\gamma)}\left[\frac{1}{\alpha_e - \alpha_m}\right]$$

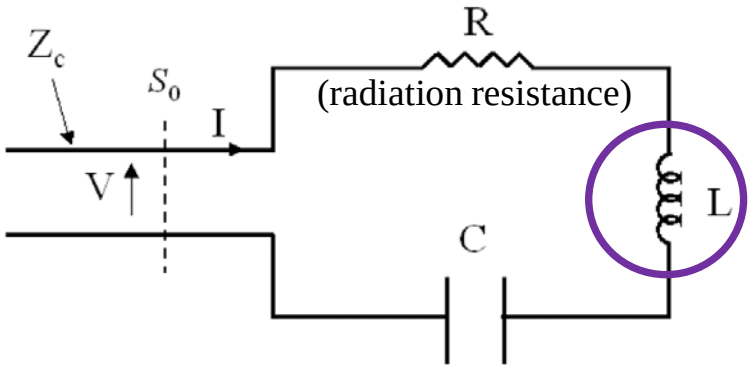
$$0 \le \gamma \le 1$$

$$Q_{lb}^{nc} = \eta/(ka)^3$$

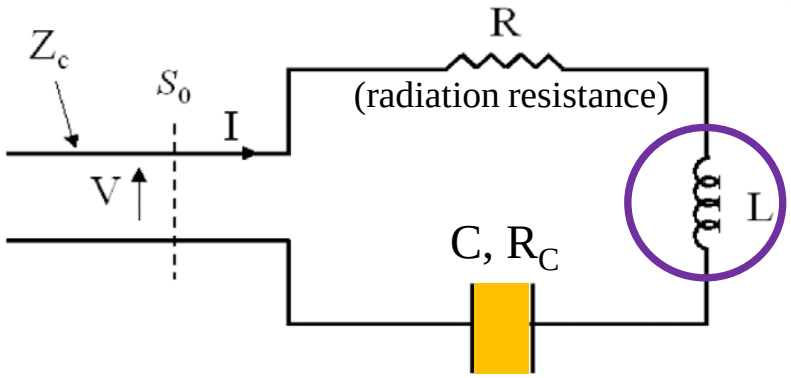
spherical electric or magnetic dipole

EXAMPLE OF MAGNETIC DIPOLE TUNED WITH CAPACITOR FILLED WITH HDL LORENTZIAN DIELECTRIC

Electrically small inductive antenna tuned with capacitor at $\omega = \omega_0$



$$\epsilon(\omega) = \epsilon_0$$



$$[\omega_0 \epsilon(\omega_0)]' = 0$$

$$\epsilon(\omega) = B\epsilon_0 \left[1 + \frac{2b^2}{1 - (\omega/\omega_0)^2 - 2ib(\omega/\omega_0)}\right]$$

$L = 28\,\mu H \quad R = 25\,\Omega \quad \omega_0 = 1/\sqrt{LC}$

$L = 28\,\mu H \quad R = 25\,\Omega \quad \omega_0 = 1/\sqrt{LC}$

$R_C = 0$
$\eta = 100\%$
$Q = 40$

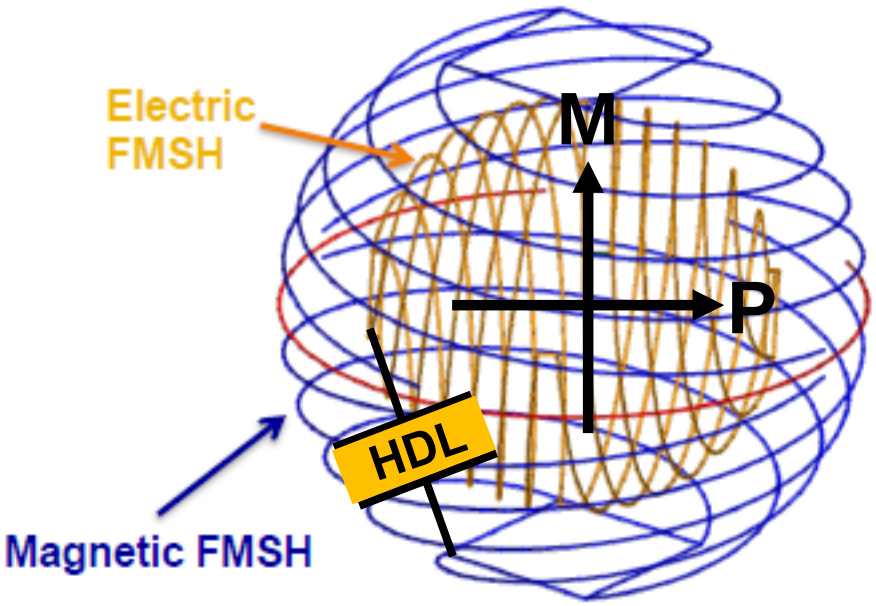
$$Q(\omega) = \eta \frac{\omega |W(\omega)|}{P_R(\omega)} \approx \frac{\omega |Z'(\omega)|}{2R(\omega)} \approx \frac{2\sqrt{\beta}}{FBW_V}$$

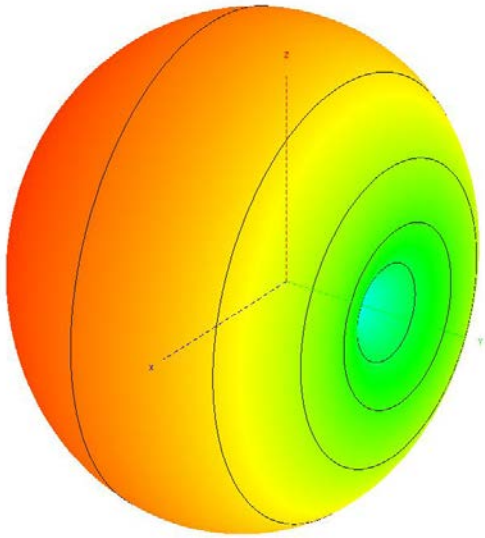
(for about a -15dB or less BW)

$R_C = 25$
$\eta = 50\%$
$Q = 10$

(Presumably, Bode-Fano matching could be also used in either case.)

HUYGENS SOURCE ESA
(Cophasal orthogonal M & P)





Directivity = 4.77 dB

For M and P to be cophasal, their currents must be 90 deg out of phase. Thal has shown that this requires a tuning capacitor or inductor that doubles the Q. An **HDL** tuning element could, in principle, achieve the 90 deg phase shift without increasing the Q, but with a decrease in the efficiency of the antenna.

SUMMARY

- Formulas for energy density and Q are generalized to hold for antennas with highly dispersive lossy material.
- These general formulas are applied to obtain lower bounds on Q of electrically small dipole antennas confined to an arbitrarily shaped volume V in terms of the PEC polarizabilities of V, the ratio of the powers radiated by the electric and magnetic dipoles, and the efficiency of the antenna.
- Highly dispersive materials enable single dipole antennas to overcome the Chu lower bound by sacrificing efficiency.
- Q lower bounds for antennas excited by global electric currents only (no magnetization) are given in the paper.